Feedback Based Dynamic Congestion Pricing

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3639 words + 3 figures + 4 tables

November 15, 2010
Abstract

This paper presents a mathematical model for dynamic congestion pricing at a toll where alternate lanes or routes are available. The model developed is based on traffic conservation law and queuing. Moreover, it uses fundamental macroscopic relationships for its derivation. The modeling uses a Logit model for the price and driver choice behavior relationship. We use this nominal mathematical model for the dynamics to derive a feedback control law that uses real-time information to determine an optimal tolling price. Simulation results demonstrate the performance of this dynamic feedback congestion pricing algorithm.
INTRODUCTION
Congestion pricing is defined as charging motorists during peak hours to encourage them to either switch their travel times or to use alternative routes which are not congested at peak hours. The theory behind road pricing suggests that, in order to reach social optimum, a toll needs to be charged that is equal to the difference between social marginal costs (which include external costs that users impose on each other on a congested road) and private average costs of users (travel delays, fuel, maintenance etc.).

In recent years, with the help of technological developments such as electronic toll collection system, pricing can also be done dynamically, that is, tolls can be set real-time varied according to the traffic conditions. Although the continuously time-varying optimal tolls suggest a fair system for the users, it is also debatable whether smoothly-varying toll rate will be appreciated by drivers. Therefore, in real world dynamic pricing applications step (piecewise constant) tolls are mainly used. Examples from US are depicted in Table 1.

<table>
<thead>
<tr>
<th>FACILITY</th>
<th>TOLLING SYSTEM</th>
<th>WEBSITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>San Diego I-15 FasTrak</td>
<td>The toll schedule varies dynamically every 6 minutes</td>
<td><a href="http://argo.sandag.org/fastrak">http://argo.sandag.org/fastrak</a></td>
</tr>
<tr>
<td>Orange County, CA SR-91</td>
<td>Toll varies every hour depending on traffic conditions</td>
<td><a href="http://www.91expresslanes.com">www.91expresslanes.com</a></td>
</tr>
<tr>
<td>Minnesota I-394</td>
<td>Tolls can be varied as frequently as 3 minutes</td>
<td><a href="http://www.mnpass.org/394/index.html">http://www.mnpass.org/394/index.html</a></td>
</tr>
</tbody>
</table>

BACKGROUND AND MOTIVATION
As mentioned in the previous section, dynamic congestion pricing application currently makes use of step functions that dynamically adjust toll rates based on the prevailing traffic conditions on the toll road. Clearly, this approach has several shortcomings including the lack of theoretical basis for the determination and implementation of tolls. Moreover, such an approach can cause unpredictable fluctuations in travel times and overall sub-optimal results in terms of users as well as the system.

In this paper, we will propose a theoretically sound feedback based congestion pricing model that will attempt to:

1. Achieve the pre-set objective such as system optimal or allowable user-equilibrium.
2. Develop a control law that is robust against uncertainties within a set.
3. Assure that the developed control law is stable and does not fluctuate in an implementable way. For example, if the dynamic toll prices change from $5 to $20 and then to $5 in a very short period of time, say 5 minutes, this will create unexpected results and low compliance rates. Thus, dynamic toll prices should increase and decrease in a relatively smooth way.
4. Incorporate bounds for maximum and minimum tolls to ensure equity.

The proposed feedback methodology is based on changing prices in real-time to ensure more efficient operations. In other words, instead of using fixed tolls, tolls are changed to reflect the prevailing traffic conditions. The failure to do so, might cause over or under utilization of either toll or non-toll lanes. Thus, the proposed methodology will not affect the total demand but will more effectively manage incoming demand by changing tolls in such a way that the system is driven to a pre-determined optimal state, which

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2http://www.tollroadsnews.com/node/1197
in this case is user optimal. On the other hand, the proposed methodology does not explicitly affect total incoming demand although one of the long-term implications of a better traffic control strategy might be changes in the total demand such that overall system delays are minimized. The pricing scheme proposed here considers the difference between marginal social and private (user) costs, without typically accounting for such externalities such as maintenance, energy consumption etc.

**LITERATURE REVIEW**

Congestion pricing has been one of the most important research topics in traffic engineering throughout the last few decades. Several studies were conducted on the theoretical aspects of pricing models ((26), (12), (7), (3), (6)). Wie (1998) (3) developed two types of dynamic congestion pricing model, based on the theory of marginal cost pricing and showed that two types of time-varying congestion tolls can be determined by solving a convex control formulation of the dynamic system optimal traffic assignment problem on a network with many origins and many destinations. Friesz et al. (2004) presented the theory of dynamic congestion pricing as a continuous time optimal control problem accomodating elastic nonseparable travel demands and nonseparable travel costs (4). Yang (1999) considered a combined route guidance and road pricing system showing a joint implementation of route guidance and road pricing in a network with recurrent congestion could drive a stochastic network flow pattern towards a system optimum, and thus achieve a higher reduction in system travel time. (10) modeled the interaction between route guidance and road pricing and evaluated the potential benefit of their joint implementation (10). A trial-and-error implementation scheme of marginal-cost pricing on a general road network when the demand functions are unknown was proposed by Yang (2004) (11). (11) proposes an iterative toll adjustment procedure based on the method of successive averages, and presented a rigorous theoretical proof of its convergence. Zhao (2006) studied an on-line, trial-and-error implementation of marginal-cost pricing for networks with users whose values of travel time vary, whose demand functions are unknown, and whose route choices conform to random-utility maximization (27). Wie (2007) studied the problem of dynamic congestion pricing that determines optimal time-varying tolls for a pre-specified subset of arcs with bottleneck on a congested general traffic network (2). (2) formulated a two-person nonzero-sum dynamic Stackelberg game model and analysed the characteristics of the Stackelberg equilibrium solution. Yang (2008) studied the day-to-day dynamic pricing schemes with the elastic demand counterpart to force the traffic system to evolve from the status quo to a stationary state of maximizing the social net benefit, considering a general drivers’ behavior adjustment process (8). (8) introduced the notion of a strong dynamic optimal toll and discussed the mathematical properties of dynamic pricing and proposed a simple solution, namely the dynamic marginal pricing scheme. Joksimovic et al. (2005) discussed the dynamic road pricing for optimizing network performance with heterogeneous users (5). In (5), the effects of time-dependent tolls on the network performance were analyzed using a dynamic traffic model. The network design problem was formulated as a bi-level optimization problem. Dimitriou (2009) studied an evolutionary game-theoretic learning model for dynamic congestion pricing in urban road networks, taking into account route choice stochasticity and reliability considerations, and the heterogeneity of users, in terms of their value of travel time and real-time information acquisition (17). In Dong et al. (13) two real-time pricing strategies are defined as alternatives to the static congestion pricing schemes. Two distinct approaches of real-time pricing namely reactive and predictive congestion pricing schemes are defined and tested using the DYNAMSMART-X, a simulation-based DTA system. While the reactive pricing strategy proposed in Dong et al. (2007) is based on the logic of the well-known feedback based control law namely ALINEA (Papageorgiou (2003) (20)), the proposed anticipatory pricing strategy uses a traffic simulator to predict future traffic conditions that are then used to set the future prices.

The main goal of our work is to formulate a non-linear dynamic model and derive a stable and robust feedback control law that will address some of the issues related to the ALINEA based “reactive” control law that Dong et al. (2007) mentioned in their paper. The main problem with using ALINEA based control law is that it is developed to optimize freeway traffic by regulating ramp flows through the use of a linearized
system dynamics model which will not be suitable for the type of dynamic pricing scenario studied in our paper. To remedy this, we first formulate a non-linear traffic model for both HOT and regular lanes and then employ feedback linearization technique to derive the feedback control law for this non-linear traffic model with the objective of driving the system to a user-equilibrium state by changing tolls dynamically. Thus, our proposed closed loop feedback control law is specifically designed to take into account traffic non-linearities to determine the tolls in real-time based on the prevailing traffic conditions.

Lindsey (2003) reviewed road pricing applications in the US and Canada by comparing the implementations in Europe and applicability of the different categories of congestion pricing to US roads (24). Other international congestion pricing experiences such as Singapore, Norway and United Kingdom were explained and lessons learned from these implementations were also analyzed by different authors ((19), (15), (25)). In practice, congestion pricing is performed generally by 1) HOT (High-occupancy toll lanes) lanes, which are the lanes reserved for vehicles that meet minimum occupant requirements or vehicles that pay tolls, 2) Cordon pricing, which is charging vehicles to access a zone (e.g. highly congested part of a metropolitan city). Some of the major road pricing applications in the US are summarized in Table 2

<table>
<thead>
<tr>
<th>TABLE 2 Major Road Pricing Initiatives in the US</th>
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<tbody>
<tr>
<td>(a) Dynamic Pricing Applications</td>
</tr>
<tr>
<td>Project/Location</td>
</tr>
<tr>
<td>(b) Time-of-Day Pricing Applications</td>
</tr>
<tr>
<td>Project/Location</td>
</tr>
</tbody>
</table>

Although congestion pricing is generally studied for simple settings such as static networks and homogeneous users, there are also several studies concerning real world conditions. Holguin-Veras and Cetin (2009) studied optimal tolls for multi-class traffic and generated analytical formulations (14). De Palma et al. (2005) analyzed time varying tolls considering departure time, route choice, mode split in a dynamic network equilibrium (1), Chen and Berstein (2004) conducted a study tolling for different user types (18). Ozbay and Yannaz-Tuzel (2008) conducted a study on the valuation of travel time and departure time choice under congestion pricing, considering the New Jersey Turnpike’s value pricing implementation (16). There have also been some recent attempts for developing real-time dynamic congestion pricing algorithms. Zhang et al. (2008) created a feedback-based tolling algorithm for high-occupancy toll lane facilities. In their algorithm, the feedback control is obtained by a step-wise function monitoring the speeds of general purpose lanes and HOT lanes and toll rates are estimated by backward calculation using logit model. Simulation results of the model showed that overall traffic conditions were improved significantly (9).
SYSTEM MODEL

Most of the real-world dynamic toll pricing projects in the US are based on the existence of a toll road and a toll free road as an alternative. Commuters are thus expected to make a decision about choosing the toll road at a decision point where the prevailing traffic conditions on both roadways as well as the current tolls are communicated to them mainly through variable message signs. Thus, each traveler makes a decision as to whether or not to pay the toll and use the toll road or to simply continue to drive on the free alternative.

A modified version of feedback routing model developed by Kachroo and Ozbay (2005, 1998a, b) with some modifications to its route cost functions, can be used as a mechanism to regulate traffic coming to the single decision model (23), (22),(21). However, it is important to first analyze that model in terms of the above requirements specific to the congestion pricing problem. Let us consider the system configuration as given in figure 1.

The symbols for the variables that will be used in the mathematical model formulation are summarized below in Table 3.

The system dynamics are given by:

\[
\begin{align*}
\ell_T & = \alpha q_{in}(t) - s_T(t) \\
\rho_T & = s_T(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
\ell_R & = (1 - \alpha) q_{in}(t) - s_R(t) \\
\rho_R & = s_R(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{align*}
\]  

The first equation gives us the rate at which the queue for the toll lane is moving. The second equation is the law of conservation for the toll lane, implying that vehicles cannot be created or destroyed in any section of the system. The next equations represent the same dynamics for the regular lane.

Using Greenshield’s fundamental relationship, which we have already used to derive equation 1, we have:

\[
\begin{align*}
q_T(t) & = v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right) \\
q_R(t) & = v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
\end{align*}
\]  

The Greenshields model will be used irrespective of queuing as it relates the flow to the density and velocity of the outgoing vehicles. Greenshields model is used to determine the flow rate in the lane which is an
TABLE 3 Symbols used in the Mathematical Formulation

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>MEANING</th>
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<tbody>
<tr>
<td>$q_{in}$</td>
<td>Traffic in-flow</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Percent flow using toll</td>
</tr>
<tr>
<td>$\ell_T$</td>
<td>Queue length for toll lane</td>
</tr>
<tr>
<td>$\ell_R$</td>
<td>Queue length for regular lane</td>
</tr>
<tr>
<td>$s_T$</td>
<td>Service rate for toll lane</td>
</tr>
<tr>
<td>$s_R$</td>
<td>Service rate for regular lane</td>
</tr>
<tr>
<td>$\rho_T$</td>
<td>Traffic density in toll lane</td>
</tr>
<tr>
<td>$\rho_R$</td>
<td>Traffic density in regular lane</td>
</tr>
<tr>
<td>$\rho_m$</td>
<td>Maximum traffic density</td>
</tr>
<tr>
<td>$v_f$</td>
<td>Traffic free flow velocity</td>
</tr>
<tr>
<td>$q_T$</td>
<td>Traffic outflow from toll lane</td>
</tr>
<tr>
<td>$q_R$</td>
<td>Traffic outflow from regular lane</td>
</tr>
<tr>
<td>$L_T$</td>
<td>Length of the toll lane</td>
</tr>
<tr>
<td>$L_R$</td>
<td>Length of the regular lane</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Fraction of toll in-flow using RF-tags</td>
</tr>
<tr>
<td>$T_T$</td>
<td>Travel time through toll lane</td>
</tr>
<tr>
<td>$T_R$</td>
<td>Travel time through regular lane</td>
</tr>
</tbody>
</table>

indication for the level of congestion. The developed feedback model uses this information to determine the tolling rate in order to maintain the “allowable” user equilibrium.

There are various modifications of the basic model in equation 1 that we can use based on the actual implementation of the tolling scheme. For instance, if the tolling is done automatically for everyone using RF-tags, then there will be no queues in the system and there will be no ETC gate. Since the queue lengths are zero, the dynamics for that implementation will have equations that are shown in equation 3

$$
\dot{\rho}_T = \alpha q_{in}(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right)
$$

$$
\dot{\rho}_R = (1 - \alpha) q_{in}(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
$$

(3)

If only the toll lane has a gate and no gate for the regular lane, then the model has equations that are shown in equation 4

$$
\dot{\ell}_T = \alpha q_{in}(t) - s_T(t)
$$

$$
\dot{\rho}_T = s_T(t) - v_f \rho_T \left(1 - \frac{\rho_T(t)}{\rho_m}\right)
$$

$$
\dot{\rho}_R = (1 - \alpha) q_{in}(t) - v_f \rho_R \left(1 - \frac{\rho_R(t)}{\rho_m}\right)
$$

(4)

If $\beta$ is the fraction of toll vehicles that use the RF-tages as depicted in Figure 2, then the dynamics will be given by equation 5
\[ \ell_T = \alpha (1 - \beta) q_{in}(t) - s_T(t) \]
\[ \dot{\rho}_T = s_T(t) + \alpha \beta q_{in}(t) - v_f \rho_T \left( 1 - \frac{\rho_T(t)}{\rho_m} \right) \]
\[ \ell_R = (1 - \alpha) q_{in}(t) - s_R(t) \]
\[ \dot{\rho}_R = s_R(t) - v_f \rho_R \left( 1 - \frac{\rho_R(t)}{\rho_m} \right) \] (5)

The most general model in our setting would allow queuing in every lane and would have the structure shown in equation 6. In this model, the queue length for tagged vehicles is given by \( \ell_{RF} \) and the service rate as \( s_{RF}(t) \).

\[ \ell_T = \alpha (1 - \beta) q_{in}(t) - s_T(t) \]
\[ \ell_{RF} = \alpha \beta q_{in}(t) - s_{RF}(t) \]
\[ \dot{\rho}_T = s_T(t) + s_{RF}(t) - v_f \rho_T \left( 1 - \frac{\rho_T(t)}{\rho_m} \right) \]
\[ \ell_R = (1 - \alpha) q_{in}(t) - s_R(t) \]
\[ \dot{\rho}_R = s_R(t) - v_f \rho_R \left( 1 - \frac{\rho_R(t)}{\rho_m} \right) \] (6)

In this model, however, if the lanes are wide enough so that no queues are formed for the tagged and regular vehicles, then we will obtain the model given by equation 7. This is the model we have used in the simulation studies presented in this paper.

\[ \ell_T = \alpha (1 - \beta) q_{in}(t) - s_T(t) \]
\[ \dot{\rho}_T = s_T(t) + \alpha \beta q_{in}(t) - v_f \rho_T \left( 1 - \frac{\rho_T(t)}{\rho_m} \right) \]
\[ \dot{\rho}_R = (1 - \alpha) q_{in}(t) - v_f \rho_R \left( 1 - \frac{\rho_R(t)}{\rho_m} \right) \] (7)

**Feedback Control Law for the Robust Congestion Pricing**

In order to derive a feedback control law for allowable user-equilibrium, we will use the formula 8 for travel time through tolled and regular lanes. A lower travel time must be maintained at the tolling lane otherwise drivers will not be willing to pay for a worse or even equal traffic conditions. Normally, user equilibrium indicates equal travel times from point A to point B. However, in the case of comparing travel times in the
tolling lane vs the regular lane an “allowable” user equilibrium is defined which maintains travel time of
tolling lane at a lower value than the regular lane. We use the symbol $\gamma$ for that factor. Therefore, when
designing the feedback control system, the error is determined by $\gamma$ where $\gamma$ must be larger than 1. The
$\gamma$ factor must take into account factors based on traffic conditions and drivers’ behavior, so that the tolling
lane would be considered worthwhile for the driver. This parameter is taken to be constant in this paper; however, it can vary based on traffic conditions as well as the parameters used in the logit model. In other
words, $\gamma$ could increase or decrease based on the desired alpha which is the percent flow using toll.

$$\begin{align*}
T_T(t) &= \ell_T(t) + \frac{L_T}{s_T(t)}v_f \left(\frac{1}{1 - \rho_T(t)}\right) \\
T_R(t) &= \ell_R(t) + \frac{L_R}{s_R(t)}v_f \left(\frac{1}{1 - \rho_R(t)}\right)
\end{align*}$$

(8)

The feedback control problem here is to maintain the allowable user-equilibrium, such that the error
defined by $\gamma$ will have closed loop dynamics that are asymptotically stable in the sense of Lyapunov. Since
the desire is that the error goes to zero in the feedback system we have time in the tolled lane $\frac{1}{\gamma}$ when
compared to the regular lane which clearly indicates a lower travel time for the tolling lane. The error is
defined by the following equation.

$$e(t) = \gamma T_T(t) - T_R(t) = \gamma \left(\frac{\ell_T(t)}{s_T(t)} + \frac{L_T}{v_f \left(1 - \frac{\rho_T(t)}{\rho_m}\right)} - \frac{\ell_R(t)}{s_R(t)} + \frac{L_R}{v_f \left(1 - \frac{\rho_R(t)}{\rho_m}\right)}\right)$$

(9)

In the above equation the goal of the control design is to make the error go to zero. We will use
feedback linearization technique to derive the feedback control law for the above non linear model. For that
design we need to differentiate the error term with respect to time. Hence, differentiating equation 9 gives
us the error dynamics as

$$\begin{align*}
\dot{e}(t) &= \gamma \dot{T}_T(t) - \dot{T}_R(t) \\
&= \gamma \left(\frac{\dot{\ell}_T(t)}{s_T(t)}\dot{s}_T(t) + \frac{\ell_T(t)}{s_T(t)}\dot{\rho}_T(t) + \frac{L_T}{v_f \rho_m \left(1 - \frac{\rho_T(t)}{\rho_m}\right)}\dot{\rho}_T(t)\right) \\
&\quad - \gamma \left(\frac{\dot{\ell}_R(t)}{s_R(t)}\dot{s}_R(t) + \frac{\ell_R(t)}{s_R(t)}\dot{\rho}_R(t) + \frac{L_R}{v_f \rho_m \left(1 - \frac{\rho_R(t)}{\rho_m}\right)}\dot{\rho}_R(t)\right)
\end{align*}$$

(10)

Expanding just one term in these error dynamics using system dynamics given by equation 5, we get...
\[\dot{T}(t) = -\frac{\ell_T(t)}{s_T^2(t)} \dot{s}_T(t) + \frac{\dot{T}(t)}{s_T(t)} + \frac{L_T}{v_f \rho_m \left(1 - \frac{T(t)}{\rho_m}\right)^2} \dot{\rho}_T(t)\]

\[= -\frac{\ell_T(t)}{s_T^2(t)} \dot{s}_T(t) + \frac{1}{s_T(t)} \left[\alpha(1 - \beta)q_{in}(t) - s_T(t)\right] + \frac{L_T}{v_f \rho_m \left(1 - \frac{T(t)}{\rho_m}\right)^2} \left[s_T(t) + \alpha \beta q_{in}(t) - v_f \rho_T(t) \left(1 - \frac{T(t)}{\rho_m}\right)\right]\]  

(11)

Similarly,

\[\dot{R}(t) = -\frac{\ell_R(t)}{s_R^2(t)} \dot{s}_R(t) + \frac{\dot{R}(t)}{s_R(t)} + \frac{L_R}{v_f \rho_m \left(1 - \frac{R(t)}{\rho_m}\right)^2} \dot{\rho}_R(t)\]

\[= -\frac{\ell_R(t)}{s_R^2(t)} \dot{s}_R(t) + \frac{1}{s_R(t)} \left[(1 - \alpha)q_{in}(t) - s_R(t)\right] + \frac{L_R}{v_f \rho_m \left(1 - \frac{R(t)}{\rho_m}\right)^2} \left[s_R(t) - v_f \rho_R(t) \left(1 - \frac{R(t)}{\rho_m}\right)\right]\]  

(12)

Substituting equations 11 and 12 into equation 10 gives us

\[\dot{e}(t) = f + g\alpha\]  

(13)

where \(f\) and \(g\) are state dependent terms whose exact formula can be extracted from using equations 11 and 12 with equation 10.

Now, we can design the feedback control law for \(\alpha\) as

\[\alpha = g^{-1}(-f - ke(t))\]  

(14)

Using this control law the closed loop error dynamics for equation 14 are asymptotically stable, that is

\[\lim_{t \to \infty} e(t) = 0\]  

(15)

Although we have obtained the closed-loop desired behavior, we still have to come up with the actual toll rate that we must charge. To come up with the functional form for that, we choose a Logit model to formulate the driver decision to choose between the tolled and regular lanes. We use the following relationship:

\[\alpha = \frac{1}{1 + \exp\left(a_1(T_T(t) - T_R(t)) + a_2 p(t) + a_3\right)}\]  

(16)

Here, \(p(t)\) is the toll rate, \(a_1\) is the weight given to the travel time difference in making a decision of taking the toll lane. In the literature, similar factors such as “Indifference band” are used to represent the minimum time difference between two routes needed for travelers to switch. More specifically, the claim is that travelers will not switch to toll lanes if the difference between toll and non-toll lanes is less than a certain amount. \(a_2\) is the weight given to the toll rate, and finally, \(a_3\) covers other factors in the driver choice.

From equation 16, we can obtain the deployable toll rate in terms of computed \(\alpha\) as
\[ p(t) = \frac{1}{a_2} \left( \ln \left( \frac{1 - \alpha}{\alpha} \right) - a_1(T_T(t) - T_R(t)) - a_3 \right) \]  

(17)

From this equation we see that if \( a_1 \) increases, which implies an increased effect of travel time in the tolled lane or if \( a_2 \) increases, which implies an increased effect of the toll rate or if \( a_3 \) increases, which implies an increased effect of other factors in the drivers choice, then the toll rate \( p(t) \) should decrease.

**SIMULATION BASED EVALUATION OF ROBUST CONGESTION PRICING**

We use Scilab software to perform simulations for the dynamics for the system given by equation 7. We use the control law given by equation 14. Now, since we are using no queues for the tagged and regular lanes, there will be no terms for the queue lengths in the controller. Moreover, for the sake of simulation we will assume that the service rate for tolling is fixed. Based on these conditions we get

\[
f = \gamma \left( \frac{-l_T(t)}{s_T} \dot{S}_T(t) + \frac{L_T}{v_f \rho_m} \left( 1 - \frac{\rho_T(t)}{\rho_m} \right)^2 \left[ S_T(t) - v_f \rho_T(t) \left( 1 - \frac{\rho_T(t)}{\rho_m} \right) \right] - 1 \right)
\]

(18)

\[
g = \gamma \left( \frac{1}{s_T} \left[ (1 - \beta)q_{in}(t) \right] + \frac{L_T}{v_f \rho_m} \left( 1 - \frac{\rho_T(t)}{\rho_m} \right)^2 \left[ \beta q_{in}(t) \right] \right) + \frac{L_R}{v_f \rho_m} \left( 1 - \frac{\rho_R(t)}{\rho_m} \right)^2 \left[ q_{in}(t) \right]
\]

(19)

The control law for the simulation is

\[
\alpha = g^{-1} \left( -f - ke(t) \right)
\]

(20)

where

\[
e(t) = \gamma \left( \frac{\ell_T(t)}{s_T} + \frac{L_T}{v_f \left( 1 - \frac{\rho_T(t)}{\rho_m} \right)} \right) - \left( \frac{L_R}{v_f \left( 1 - \frac{\rho_R(t)}{\rho_m} \right)} \right)
\]

(21)

The simulation parameters are given in Table 4.

We use a variable inflow rate to see how the system would evolve. The feedback control law tries to keep the error rate low. In the simulation we also make sure that the implemented value of the split is between zero and one, and also that the queue length and all other state variables always remain non-negative. The simulation results are shown in Figure 3.

The traffic inflow for the simulations is given by: \( q_{in} = 1000(1 + 0.25 \sin(0.5t) + 0.125 \sin(t)) \) which tests the robustness of the developed control methodology by varying the inflow rate. As depicted in Figure 3, when the split or the control variable \( \alpha \) increases, that is more vehicles are desired to utilize the toll lane, the density in the regular lane decreases as shown in the density plot for the regular lane. Furthermore, the error is going to zero as demonstrated in the error plot even though our total inflow \( q_{in} \) is fluctuating based on how it is defined.
FIGURE 3 Feedback Tolling Results
### TABLE 4 Simulation Parameters

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>VALUE</th>
</tr>
</thead>
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<td>$k$</td>
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#### CONCLUSIONS

In this paper we formulated the mathematical models for different scenarios of tolling. The models allowed the flexibility for having RF tags based lanes and also regular lanes. The paper showed how to build models in a modular fashion to include the needed features. The models were then used to design real-time feedback controller using feedback linearization technique to regulate the traffic in the different lanes (or routes). Simulation software was developed using Scilab to show the robustness and performance of the algorithm, and it provided the validation for the control design.

#### REFERENCES


